1 Lévy flights and continuous time random walks (CTRW)

1.1 Random walks and Lévy flights

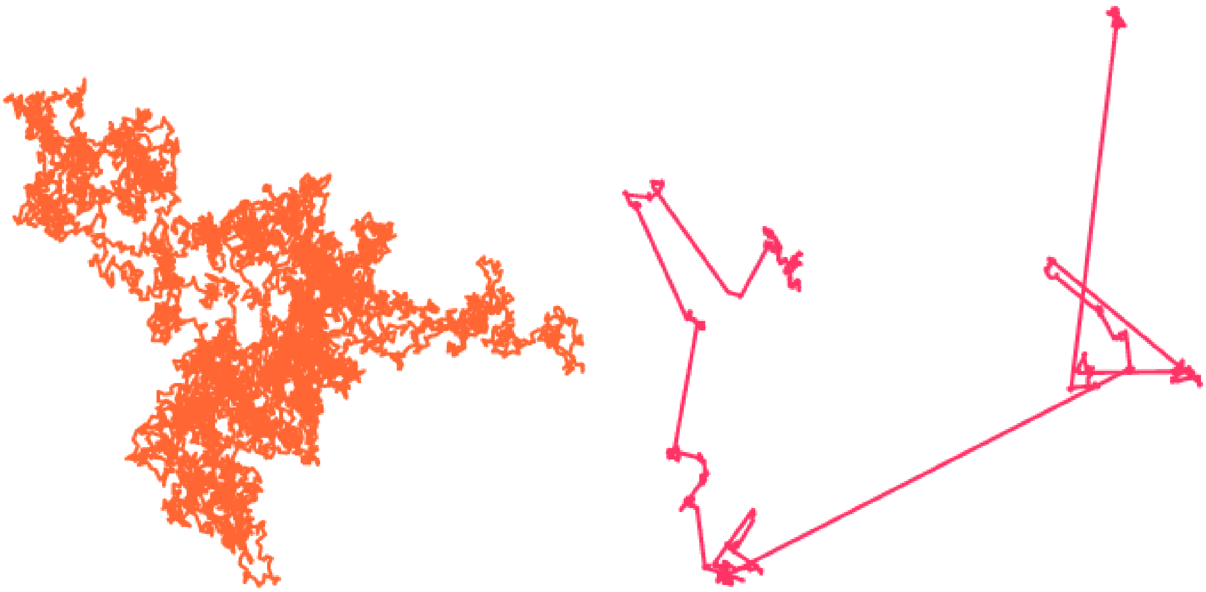

Figure 1.1: Two dimensional trajectories of random walks on large scales. Left: Ordinary random walk with finite, well defined variance of single steps. Right: A Lévy flight trajectory of index $\beta = 1$.

The position of an ordinary random walk is frequently defined as a sum of N independent identically distributed displacements ΔX_n:

$$X_N = \sum_{n=1}^{N} \Delta X_n. \tag{1.1}$$

Each displacement is drawn from the same probability density function (pdf) $p(\Delta x)$. Here it suffices to discuss symmetric single step pdfs in one dimension. According to the central limit theorem the pdf $W_Y(y, N)$ for the scaled position

$$Y_N = \frac{X_N}{\sqrt{N}} \tag{1.2}$$

is independent of N in the limit $N \to \infty$ and Gaussian, i.e.

$$\lim_{N\to\infty} W_Y(y, N) = W_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}, \tag{1.3}$$

where $\sigma^2$is the variance of the single steps ΔX_n. From Eq. (1.2) one can read off the universal scaling relation

$$X_N \sim \sqrt{N} \tag{1.4}$$

for ordinary random walks. Alternatively one can compute the variance of X_N as a function of step number N,

$$\langle X_N^2 \rangle = \sigma^2 N. \tag{1.5}$$

Eq. (1.2) in combination with Eq. (1.3) implies that for large N the pdf $W_X(x, N)$ for the position X_N is asymptotically a spreading Gaussian:

$$W_X(x, N) \sim \frac{1}{\sqrt{N}} W_Y(x/\sqrt{N}). \tag{1.6}$$

The fact that $W_X(x, N)$ depends on the ratio $x/\sqrt{N}$ merely is another way of stating the scaling relation (1.4). Note however, that for relation (1.6) the existence of the variance of the single steps is no requirement as opposed to Eq. (1.5). On large scales trajectories of ordinary random walks resemble ordinary Brownian motion.

Lévy flights belong to a class of random walks for which the central limit theorem does not apply. They can be defined in a similar fashion as ordinary random walks, i.e. by a sum of independent identically distributed random increments (Eq. (1.1)). If the single step pdfs possess algebraic tails however, such that the single step second moment is divergent, i.e.

$$p(\Delta x) \sim \frac{1}{\Delta x^{1+\beta}} \qquad 0 < \beta < 2, \tag{1.7}$$

a generalization of the central limit theorem, the Lévy Khinchin theorem applies. It states that, if the position of a Lévy flight is scales according to

$$Y_N = \frac{X_N}{N^{1/\beta}}, \tag{1.8}$$

the scaled variable possesses a pdf independent of N in the limit $N \to \infty$, i.e.

$$\lim_{N\to\infty} W_{Y,\beta}(y, N) = W_{Y,\beta}(y). \tag{1.9}$$

The limiting density $W_{Y,\beta}(y)$ is referred to as a Lévy stable law of index β and is no longer Gaussian. It can be expressed most easily in Fourier-space:

$$W_{Y,\beta}(y) = \frac{1}{2\pi} \int \mathrm{d}k \, e^{-iky - D|k|^\beta}, \tag{1.10}$$

where D is some constant. Asymptotically, the limiting density has the same power law behaviour as the single step distribution,

$$W_{Y,\beta}(y) \sim \frac{1}{|y|^{1+\beta}}.$$

Combining Eqs. (1.8) and (1.10) one can obtain an explicit expression for the pdf of X_N in the limit of large step number,

$$W_{X,\beta}(x, N) \sim \frac{1}{N^{1/\beta}} W_{Y,\beta}\left(x/N^{1/\beta}\right).$$

This implies that the position of a Lévy flight scales superdiffusively with the step number:

$$X_N \sim N^{1/\beta}.$$

Geometrically, trajectories of Lévy flights are easily distinguished from those of ordinary Brownian motion. In Fig. 1.1 a two-dimensional trajectory of a Lévy flight is compared with a trajectory of ordinary Brownian motion.

1.2 Continuous time random walks

Temporally continuous random walks can be easily constructed from time discrete random walks by identifying the step number N with the time elapsed t and the associated time increment $\Delta t = t/N$ between successive steps. A generalization of this concept is the continuous time random walk (CTRW), a simple version of which is defined by two pdfs: one for the spatial displacements $f(\Delta x)$ and one for random temporal increments $\phi(\Delta t)$. The CTRW then consist of pairwise random and stochastically independent events, a spatial displacement Δx and a temporal increment Δt drawn from the combined pdf

$$p(\Delta x, \Delta t) = f(\Delta x)\phi(\Delta t).$$

After N iterations the position of the walker is given by

$$X_N = \sum_{n=1}^{N} \Delta x_n$$

and the time elapsed is

$$T_N = \sum_{n=1}^{N} \Delta t_n.$$

The quantity of interest is the position $X(t)$ after time t. The pdf $W(x,t)$ for this process can be computed in a straightforward fashion [1] and can be expressed in terms of the pdfs $f(\Delta x)$ and $\phi(\Delta t)$. The Fourier-Laplace transform of $W(x,t)$ is given by

$$W(k,u) = \frac{1-\phi(u)}{u\,(1-\phi(u)\,f(k))}, \tag{1.11}$$

where $\phi(u)$ and $f(k)$ denote the Laplace- and Fourier transform of $\phi(\Delta t)$ and $f(\Delta x)$, respectively. The pdf $W(x,t)$ is then obtained by inverse Laplace-Fourier transform

$$W(x,t) = \frac{1}{2\pi}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}u \int \mathrm{d}k\, e^{ut-ikx} W(k,u). \tag{1.12}$$

$W(x,t)$ may exhibit four different universal behaviours which only depend on the asymptotics of $f(\Delta x)$ and $\phi(\Delta t)$ and thus the behaviour of $f(k)$ and $\phi(u)$ for small arguments.

1.2.0.1 Ordinary Diffusion

When both, the variance of the spatial steps and the expectation value of the temporal increments exist the Fourier- and Laplace transform of $f(\Delta x)$ and $\phi(\Delta t)$ read

$$\begin{aligned} f(k) &= 1-\sigma^2 k^2 + \mathcal{O}(k^4) \\ \phi(u) &= 1-\tau u + \mathcal{O}(u^2), \end{aligned}$$

where σ^2 and τ are some constants. Inserted into Eq. (1.11) and employing inversion (1.12) one obtains asymptotically

$$W(x,t) \sim \frac{1}{\sqrt{t}} e^{-x^2/Dt}.$$

Thus, CTRW are equivalent to Brownian motion on large spatio-temporal scales.

1.2.0.2 Lévy Flights

When the spatial displacements are drawn from a power-law pdf such as (1.7) the Fourier transform for small arguments is given by

$$f(k) = 1 - D_\beta |k|^\beta + \mathcal{O}(k^2).$$

When combined with temporal increments with finite expectation value, the same procedure as outlined above yields

$$W(x,t) \sim \frac{1}{t^{1/\beta}} L_\beta(x/t^{1/\beta}),$$

where L_β is a Lévy stable law of index β. Consequently, a CTRW with algebraically distributed spatial steps of infinite variance is equivalent to ordinary Lévy flights with a superdiffusive scaling with time

$$X(t) \sim t^{1/\beta}.$$

1.2.0.3 Fractional Brownian motion (subdiffusion)

The complementary scenario occurs when ordinary spatial steps (finite variance and $f(k) \approx 1 - \sigma^2 k^2$) are combined with a power-law in the pdf for temporal increments:

$$\phi(\Delta t) \sim \frac{1}{\Delta t^{1+\alpha}} \quad 0 < \alpha < 1.$$

In this case, the time between successive spatial increments can be very long, effectively slowing down the random walk. The Laplace transform for $\phi(\Delta t)$ is given by

$$\phi(u) = 1 - D_\alpha u^\alpha,$$

where D_α is some constant. One obtains for the position of such a random walk

$$W(x,t) = \frac{1}{2\pi} \int \mathrm{d}k e^{-ikx} E_\alpha(-D_\alpha k^2 t^\alpha), \tag{1.13}$$

where the function $E_\alpha$is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}.$$

It is easily checked that

$$W(x,t) \sim \frac{1}{t^{\alpha/2}} G_\alpha(x/t^{\alpha/2}),$$

where G_α is a non-Gaussian limiting function. From this the scaling relation

$$X(t) \sim t^{\alpha/2}$$

can be obtained. Since $\alpha < 1$ these processes are subdiffusive and sometimes referred to as fractional Brownian motion.

1.2.0.4 Ambivalent processes

The last and most interesting combination of waiting times and spatial steps is the one in which long waiting times compete and interfere with long range spatial steps, i.e. if both $\phi(\Delta t)$ and $f(\Delta x)$ decay asymptotically as a powerlaw:

$$f(\Delta x) \quad \sim \quad \frac{1}{\Delta x^{1+\beta}} \qquad 0 < \beta < 2$$

and

$$\phi(\Delta t) \sim \frac{1}{\Delta t^{1+\alpha}} \qquad 0 < \alpha < 1.$$

In this case

$$\begin{aligned} f(k) &= 1 - D_\beta |k|^\beta + \mathcal{O}(k^2) \\ \phi(u) &= 1 - D_\alpha u^\alpha + \mathcal{O}(u^2). \end{aligned}$$

The asymptotic pdf for the position of the ambivalent process can again be expressed in terms of a Fourier inversion and the Mittag-Leffler function according to

$$W(x,t) = \frac{1}{2\pi} \int \mathrm{d}k e^{-ikx} E_\alpha(-D_\alpha |k|^\beta t^\alpha). \tag{1.14}$$

Note, however, the term $|k|^\beta$in the argument of E_α. From Eq. (1.14) one can extract the scaling relation

$$X(t) \sim t^{\alpha/\beta}.$$

The ratio of the exponents α/β resembles the interplay between sub- and superdiffusion. For $\beta < 2\alpha$ the ambivalent CTRW is effectively superdiffusive, for $\beta > 2\alpha$ effectively subdiffusive. For $\beta = 2\alpha$ the process exhibits the same scaling as ordinary Brownian motion, despite the crucial difference of infinite moments and a non-Gaussian shape of the pdf $W(x,t)$.

The various types of asymptotic universal behaviours are depicted in Fig. 1.2 which shows a phase diagram spanned by the temporal exponent α and the spatial exponent β.

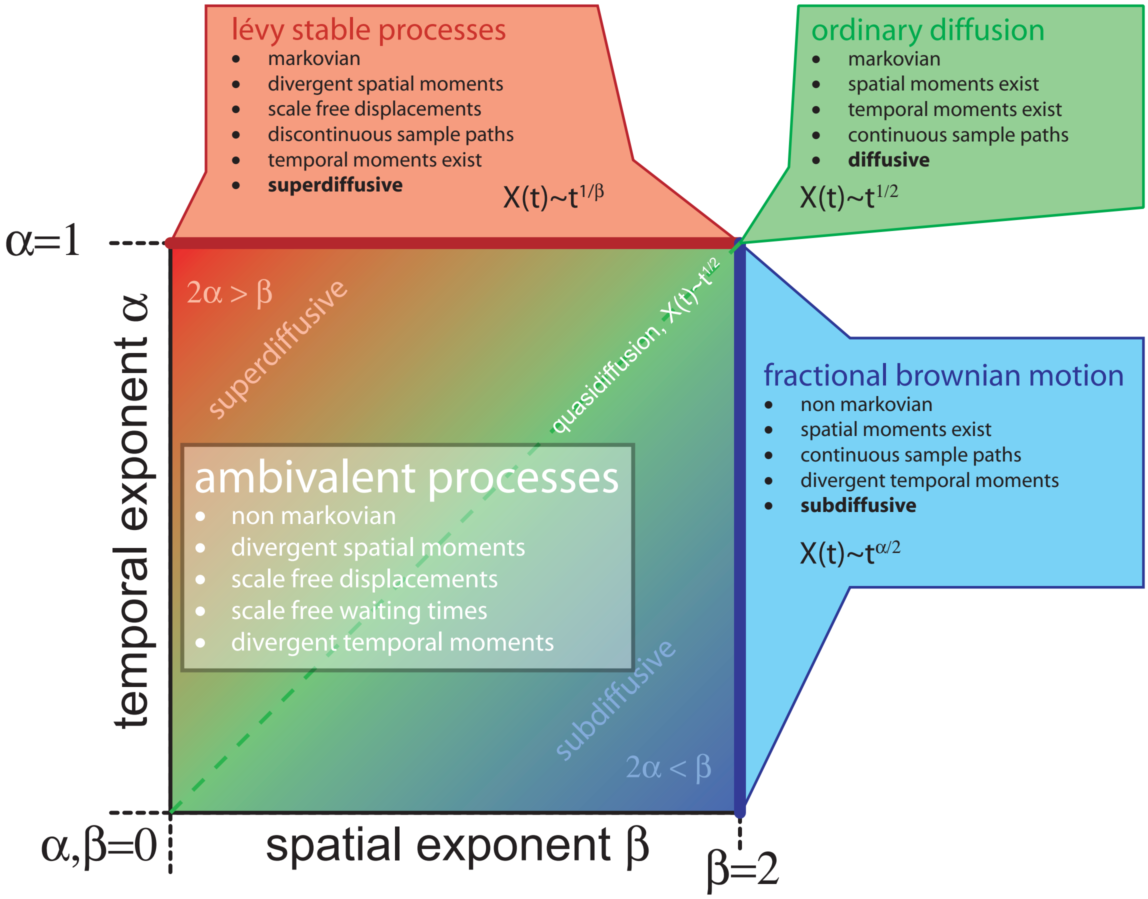

Figure 1.2: The asymptotic universality classes of continuous time random walks defined in the text as a function of the universality exponents $0 < \alpha < 1$ and $0 < \beta < 2$. Lévy flights, fractional Brownian motion as well as ordinary diffusion are limiting cases of the more general class of ambivalent processes.

Bibliography

[1] Ralf Metzler and Joseph Klafter, *The random walks guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), 1–77.